Boundary Conditions for the Continuous Sedimentation of Ideal Suspensions

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In a recent work, Bustos et al. (1990a) presented a mathematical model for sedimentation of ideal suspensions in a continuous thickener as an initial value problem. They studied the transient states by constructing corresponding global weak solutions. This formulation did not allow the control of the operation, since each change in the control variable had to be analyzed as a sequence of initial value problems, implying the necessity of imposing different values for the variable at the feeding level. To include such an analysis it is necessary to consider the continuous sedimentation process as an initial and boundary value problem (Bustos, et al., 1990b).

In his work on the continuous sedimentation of a suspension with nonconvex flux-density function, Petty (1975) analyzes the possible boundary conditions for the transient operation of a continuous sedimentation column from a physical point of view. He recognized that there are no unique values for the conditions at the feeding and discharge level. For the discharge level, he proposes the boundary condition as:

$$f'[\varphi(0,t)] \leq 0 \text{ and } \varphi_D = f[\varphi(0,t)]/q, \tag{1}$$

where φ is the concentration of the suspension, expressed as volume fraction of solids, φ_D is the discharge concentration, $f<0$ is the solid flux-density function defined as the product of the solid concentration and its velocity, and $q<0$ is the volume-average velocity. He states that this inequality expresses the physical idea that solid does not accumulate in the thickener because of imperfections in the removal mechanism and the fact that the flux-density, not the solid velocity, is continuous across the discharge level. For the feeding level $z=L$, he writes:

$$f'[\varphi(L,t)] < 0 \text{ and } f[\varphi(L,t)] = Q_F\varphi_F/S, \tag{2}$$

where Q_F and φ_F are the volume feed rate and suspension feed concentration, respectively, and S is the cross-sectional area of the thickener. The author states that the multiplicity of concentrations at the feeding level is not necessarily incongruous with physical possibilities and that the apparent ambiguity arises by supressing the dynamics associated with splitting the feed into an underflow and overflow component.

The existence and uniqueness of the solution for one conservation law with initial and boundary conditions have been studied by several authors in the last decade (Le Roux, 1981; Debois and Floch, 1988; Bardos et al., 1979; Bustos et al., 1990b). It is the purpose of this work to apply these results to the continuous sedimentation of ideal suspensions in a thickener and to show that the values assumed by the concentration at the boundaries $z=0$ and $z=L$ are more restricted than those established by Petty.

Continuous Sedimentation as an Initial and Boundary Value Problem

The gravity sedimentation of an ideal suspension in an ideal continuous thickener (ICT) may be described with the following field variables (Bustos et al., 1990a): the volume fraction of solids $\varphi(z,t)$ and the solid flux-density $f[\varphi(z,t)]$. They constitute a *continuous Kynch sedimentation process* (CKSP), if in the domain $Q_T = [(z,t) | 0<z<L, 0<t<T]$, they obey the following equations:

$$\frac{\partial\varphi}{\partial t} + \frac{\partial f(\varphi)}{\partial z} = 0, \quad (z,t)\in Q_T, \tag{3}$$

$$\varphi(z,0) = \varphi_o(z), \quad z\in(0,L), \tag{4}$$

$$\left.\begin{array}{l}\varphi(0,t) = \varphi_\infty,\\ \varphi(L,t) = \varphi_L(t),\end{array}\right\} t\in(0,T), \tag{5}$$

where f is a continuous function with continuous first derivative, defined by $f(\varphi) = q\varphi + f_b(\varphi)$; $q\leq 0$, a constant in Q_T, is the volume-average velocity of the mixture defined by $q = \varphi v_s + (1-\varphi)v_f$; $f_b \leq 0$ is the batch solid flux-density function; $v_s<0$ and v_f are the velocities of the solid and fluid component, respectively, and T is a positive real number; φ_∞, the final concentration in batch sedimentation, is a constant, and $\varphi_o(z)$ and $\varphi_L(t)$ are piecewise constant functions representing the initial concentration profile and the concentration at the boundary $z=L$, respectively.

The kinematical process is determined, once a constitutive equation has been postulated for the flux-density function, f_b, in terms of the concentration and the motion.

The solution to *the initial value problem* represented by the nonlinear conservation law (Eq. 3) and the initial condition (Eq. 4) is, in general, only piecewise continuous, having a finite number of discontinuities $z(t)$. Across these discontinuities, the solutions satisfy the Rankine-Hugoniot conditions:

$$\frac{dz(t)}{dt} = \delta(\varphi^+,\varphi^-) = \frac{f(\varphi^+) - f(\varphi^-)}{\varphi^+ - \varphi^-}, \tag{6}$$

where δ is the speed of propagation of the discontinuity and $\varphi^{\pm} = \varphi[z(t) \pm 0,t]$.

These solutions, called *generalized solutions* (Lax, 1957; Kruskov, 1970), are not unique, and an additional requirement, called *entropy condition*, is needed to select the physically relevant solution (Oleinik, 1957). This requirement is:

$$\frac{f(\varphi) - f(\varphi^-)}{\varphi - \varphi^-} \leq \frac{f(\varphi^+) - f(\varphi^-)}{\varphi^+ - \varphi^-}, \tag{7}$$

for all φ between φ^+ and φ^-. Geometrically this requirement implies that the flux-density function stays on one side of the chord that joints the points $[\varphi^+, f(\varphi^+)]$ with $[\varphi^-, f(\varphi^-)]$ in a flux-density vs. concentration plot. A discontinuity that satisfies inequality (Eq. 7) is called a *shock*, and a generalized solution in which discontinuities are shocks is called an *entropy solution*.

If we now consider the *initial and boundary value problem*, represented by Eqs. 3 to 5, an *entropy condition on the boundaries* $z=0$ and $z=L$ must be defined to obtain uniqueness in addition to the entropy condition (Eq. 7) defined in Q_T (Le Roux, 1981). It has been shown that this entropy condition at the boundaries $z=0$ and $z=L$ is given, respectively, by (Bustos et al., 1990b):

$$sg[\gamma\varphi(0,t) - k]\{f[\gamma\varphi(0,t)] - f(k)\} \leq 0, \tag{8}$$

for all k between $\gamma\varphi(0,t)$ and φ_∞, and by:

$$sg[\gamma\varphi(L,t) - k]\{f[\gamma\varphi(L,t)] - f(k)\} \geq 0, \tag{9}$$

for all k between $\gamma\varphi(L,t)$ and $\gamma\varphi_L(t)$. Here, sg is the sign function, and $\gamma\varphi(0, t)$ and $\gamma\varphi(L,t)$ denote the *values assumed* by the entropy solution at the boundaries $z=0$ and $z=L$, respectively. This last statement makes clear that the entropy solution to the problem (Eqs. 3 to 5) will *not necessarily satisfy* the boundary conditions (Eq. 5). Hence, a new definition of boundary condition is needed to have a well posed problem.

Admissible Values for the Concentration at Boundaries

It is a fact that for a continuous sedimentation process we cannot have an arbitrary value of the concentration as boundary condition for $z=0$ and $z=L$. Therefore, it is necessary to define, for each CKSP, the sets $\epsilon_o[\varphi_\infty]$ and $\epsilon_L[\varphi_L(t)]$ of admissible concentration values at the boundaries as those values of φ that satisfy, for every k between φ and φ_∞, the inequality (Dubois and Le Floch, 1988):

$$\frac{f(\varphi) - f(k)}{\varphi - k} \leq 0 \text{ at } z=0 \tag{10}$$

and for every k between φ and $\varphi_L(t)$ the inequality:

$$\frac{f(\varphi) - f(k)}{\varphi - k} \geq 0 \text{ at } z=L. \tag{11}$$

With these definitions, we can reformulate the problem (Eqs. 3 to 5) in the following way: *the functions $\varphi(z, t)$ and $f(z, t)$ constitute a continuous Kynch sedimentation process if they satisfy the set of equations:*

$$\frac{\partial\varphi}{\partial t} + \frac{\partial f(\varphi)}{\partial z} = 0, \ (z,t)\in Q_T \tag{12}$$

$$\varphi(z,0) = \varphi_o(z), \ z\in(0,L), \tag{13}$$

$$\gamma\varphi(0,t)\in\epsilon_o[\varphi_\infty], \ 0<t, \tag{14a}$$

$$\gamma\varphi(L,t)\in\epsilon_L[(\varphi_L(t)], \ 0<t. \tag{14b}$$

Geometrical Interpretation of the Sets $\epsilon_o(\varphi_\infty)$ and $\epsilon_L(\varphi_L)$

Consider first condition 10 at the boundary $z=0$. This condition indicates that for an arbitrary flux-density function, the chord that joins the points $[\varphi, f(\varphi)]$ and $[k, f(k)]$ in a flux-density function vs. concentration plot *is nonpositive*, see Figure 1. It is interesting to note that Petty (1975) recognized this problem and by pure physical arguments indicates that at $z=0$, $f'(\varphi)\leq 0$, which is a necessary, but not sufficient, condition for a concentration to belong to the set $\epsilon_o[\varphi_\infty]$.

At $z=L$ condition 11 indicates that the chord joining the points $[\varphi, f(\varphi)]$ and $[k, f(k)]$ in a flux-density function vs. concentration plot is *nonnegative*. Figure 2 shows boundary conditions for a specific case of a flux-density function with one inflexion point.

Consider a CKSP represented by Eqs. 12 to 14 and a flux-density function with one inflexion point at φ_a and with $f'(\varphi_\infty)<0$. As initial condition we take:

$$\varphi_o(z) = \begin{cases} \varphi_\infty, 0\leq z\leq c, \\ \varphi_L, c\leq z\leq L, \end{cases} \tag{15}$$

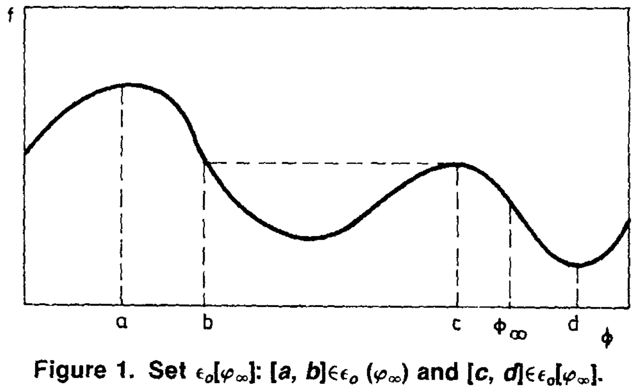

Figure 1. Set $\epsilon_o[\varphi_\infty]$: $[a, b]\in\epsilon_o(\varphi_\infty)$ and $[c, d]\in\epsilon_o[\varphi_\infty]$.

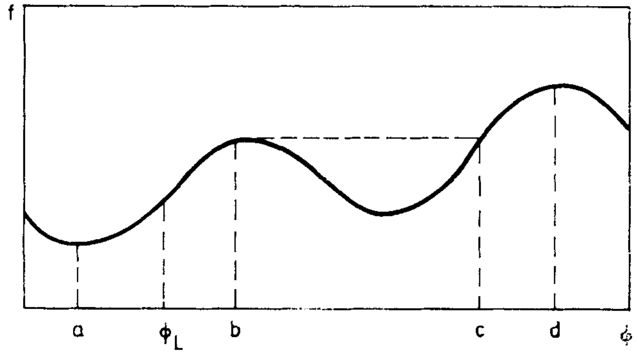

Figure 2. Set $\epsilon_L[\varphi_L]$: $[a, b] \in \epsilon_L[\varphi_L(t)]$ and $[c, d] \in \epsilon_L[\varphi_L(t)]$.

where $c<L$ and φ_L are positive constants. To construct the solution we use the method of characteristics (Bustos, et al., 1990a), so that it satisfies:

- Differential equation (Eq. 12) at the points of continuities
- Rankine-Hugoniot condition (Eq. 6) and the entropy condition (Eq. 7) across a shock
- Initial condition (Eq. 15)
- Boundary entropy conditions (Eqs. 8 and 9).

$\varphi_M < \varphi_\infty$ denotes the point at which the flux-density function f attains its relative maximum, and $\varphi_N < \varphi_M$ is a concentration such that $f(\varphi_N) = f(\varphi_M)$. Choose a concentration $\varphi_L > \varphi_N$ and denote by φ_L^* the concentration of the point of tangency of a line drawn from point φ_L to the flux-density curve, as shown in Figure 3. Let $\varphi_{Lo} > \varphi_L$ be such that $f(\varphi_{Lo}) = f(\varphi_L)$. The entropy solution to the problem is shown in the z vs. t plot of Figure 4, where

$$z_1(t) = c + f'(\varphi_\infty)t, \; 0 \le t < t_1, \tag{16}$$

$$z_2(t) = c + f'(\varphi_L^*)t, \; 0 \le t \le t_2, \tag{17}$$

and t_1 and t_2 are the following positive numbers:

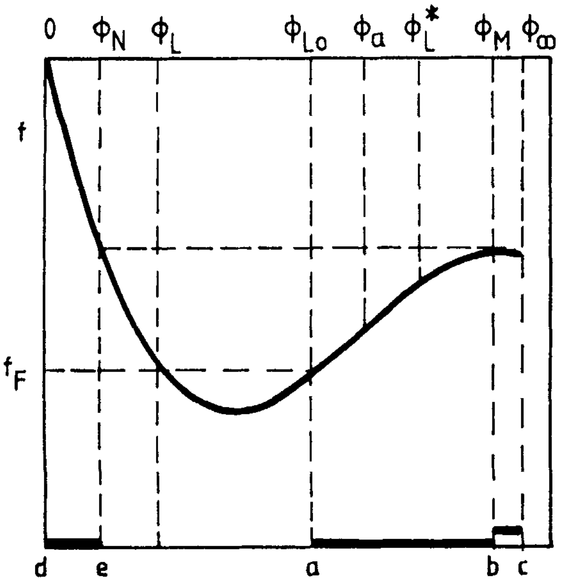

Figure 3. Flux-density function with one inflexion point showing the sets $\epsilon_o[\varphi_\infty] = [d, e] \cup [b, c]$ and $\epsilon_L[\varphi_L, t] = \{\varphi_L\} \cup [a, b]$, where $f_F = f(\varphi_L)$.

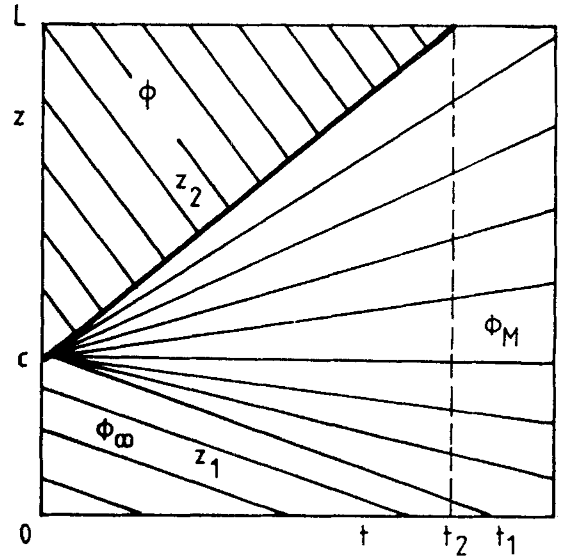

Figure 4. Entropy solution for a CKSP with a flux density function with one inflexion point and $\varphi_L > \varphi_N$.

$$t_1 = -\frac{c}{f'(\varphi_\infty)}, \tag{18}$$

$$t_2 = \frac{L-c}{f'(\varphi_L^*)} \equiv -\frac{L-c}{\delta(\varphi_L, \varphi_L^*)}. \tag{19}$$

The sets $\epsilon_o[\varphi_\infty]$ and $\epsilon_L[\varphi_L]$ are:

$$\epsilon_o[\varphi_\infty] = [0, \varphi_N] \cup [\varphi_M, \varphi_\infty], \tag{20}$$

$$\epsilon_L[\varphi_L, t] = \{\varphi_L\} \cup [\varphi_{Lo}, \varphi_M]. \tag{21}$$

By simple inspection we can see that the solution to the problem (Eqs. 12 through 14), constructed in this way, satisfies the differential equation (Eq. 12) at the points of continuity, the Rankine-Hugoniot and entropy conditions (Eqs. 6 and 7), respectively, across $z_2(t)$ and the initial condition (Eq. 15). We still have to prove that it does satisfy the boundary entropy condition (Eqs. 8 and 9).

The line $z_1(t)$ is the line of continuity limiting the region of maximum concentration φ_∞ with a rarefaction wave spanning the concentration range from φ_∞ to φ_M. Therefore, the values assumed by the solution $\varphi(z, t)$ at $z=0$ for $t > t_1$ belong to the set $\epsilon_o[\varphi_\infty]$. The line $z_2(t)$ is a contact discontinuity limiting the constant concentration φ_L with the rarefaction wave spanning the concentrations from φ_L^* to φ_M. Then, the values assumed by the solution at $z=L$ for $t > t_2$ belong to the set $\epsilon_o[\varphi_L(t)]$.

It is clear then that even though the solution does not satisfy the classical boundary conditions for all t, it does satisfy the generalized boundary conditions and therefore it is the unique entropy solution to the problem.

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Notation

c = positive real number
f = continuous solid-flux density function
f_b = batch solid flux-density function
k = real number
L = height of the feeding level
q = volume average velocity
Q_F = volume feed rate
S = thickener crossectional area
sg = sign function
t = time
t_1 = positive real number
t_2 = positive real number
T = positive real number
v_s = velocity of the solid component
v_f = velocity of the fluid component
$z_1(t)$ = line of continuity
$z_2(t)$ = contact discontinuity

Greek letters

$\varphi(z, t)$ = concentration expressed as volume fraction of solids
φ_a = concentration at the inflexion point
φ_D = discharge concentration
φ_F = feed concentration
$\varphi_L(t)$ = concentration at $z=L$
φ_L^* = concentration at the point of tangency from $[\varphi_L, f(\varphi_L)]$ to the flux-density function in a f vs. φ plot
φ_{Lo} = concentration such that $f(\varphi_{Lo})=f(\varphi_L)$
φ_M = concentration at which f attains a maximum
φ_N = concentration such that $f(\varphi_N)=f(\varphi_M)$
φ_∞ = final concentration in batch sedimentation
$\varphi_o(z,t)$ = initial concentration
$\varphi^\pm$ = values of the concentration at each side of a discontinuity
δ = speed of propagation of a shock
$\gamma\varphi(0,t)$ = value assumed by the concentration at $z=0$
$\gamma\varphi(L,t)$ = value assumed by the concentration at $z=L$
$\epsilon_o[\varphi_\infty]$ = set of admissible concentrations at $z=0$
$\epsilon_L[\varphi_L(t)]$ = set of admissible concentrations at $z=L$

Literature Cited

Bardos, C., A. Y. Le Roux, and J. C. Nedelec, "First-Order Quasilinear Equations with Boundary Conditions," *Comm. Partial Differential Equations*, **4**, 1017 (1979).

Bustos, M. C., F. Concha, and W. Wendland, "Global Weak Solutions to the Problem of Continuous Sedimentation of an Ideal Suspension," *Math. Meth. in Appl. Sci.*, **13**, 1 (1990a).

Bustos, M. C., F. Paiva, and W. Wendland, "Control of Continuous Sedimentation of Ideal Suspensions as an Initial and Boundary Value Problem," *Math. Meth. in Appl. Sci.*, **12**, 533 (1990b).

Dubois, F., and P. Le Floch, "Boundary Conditions for Nonlinear Hyperbolic Systems of Conservation Laws," *J. Differential Equations*, **71**, 93 (1988).

Kruzkov, S. N., "First-Order Quasilinear Equations in Several Independent Variables," *Math USSR-Sb.*, in English, **10**, 217 (1970).

Lax, P. D., "Hyperbolic Systems of Conservation Laws: II," *Comm. Pure Appl. Math.*, **10**, 537 (1957).

Le Roux, A. Y., "Convergence of an Accurate Scheme for First-Order Quasilinear Equations," *R.A.I.R.O. Analyse Numerique*, **15**, 151 (1981).

Oleinik, O. A., "Discontinuous Solutions of Nonlinear Differential Equations," *Amer. Math. Soc. Trans. Serv.*, **26**, 95 (1957).

Petty, C. A., "Continuous Sedimentation of a Suspension with a Nonconvex Flux Law," *Chem. Eng. Sci.*, **30**, 1451 (1975).

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